# Rutgers University: Algebra Written Qualifying Exam

## August 2018: Problem 3 Solution

**Exercise.** Let $M$ be a square matrix with complex coefficients. We consider the usual matrix exponential

$$\exp(M) = \sum_{j=0}^{\infty} \frac{1}{j!} M^j.$$

Prove that $\exp(M)$ is equal to the identity matrix if and only if $M$ is diagonalizable with eigenvalues in $2\pi\mathbb{Z}$.

Solution.

If $\lambda$ is an **eigenvalue** of $M$ then $\exists \vec{v}$ such that $M\vec{v} = \lambda\vec{v}$

$$\begin{aligned} \implies \exp(M)\vec{v} &= \left( \sum_{j=0}^{\infty} \frac{1}{j!} M^j \right) \vec{v} \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} M^j \vec{v} \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \lambda^j \vec{v} \\ &= e^{\lambda}\vec{v} \end{aligned}$$

Thus, $e^{\lambda}$ is an eigenvalue of $\exp(M)$, and $\vec{v}$ is the corresponding eigenvector.

Also, $e^{\lambda} = 1 \iff \in 2\pi\mathbb{Z}$.

Therefore, the eigenvalues of $\exp(M)$ are all 1 $\iff$ $M$ has eigenvalues in $2\pi\mathbb{Z}$.

Also, $M$ is diagonalizable $\iff$ $q_M(x)$ has simple roots
$\iff$ the JCF of $M$ has Jordan blocks of size 1
$\iff$ $M$ has $n$ Jordan blocks

Thus,

$M$ is diagonalizable with eigenvalues in $2\pi\mathbb{Z}$
$\iff$ $M$ has eigenvalues $\{\lambda_1 . \dots , \lambda_m\} \subseteq 2\pi\mathbb{Z}$ and $M$ has $n$ Jordan blocks
$\iff$ $M$ has eigenvalues $\{\lambda_1 . \dots , \lambda_m\} \subseteq 2\pi\mathbb{Z}$ and $\sum_{i=1}^{m} \dim(\lambda_i\text{-eigenspace}) = n$
$\iff$ $M$ has eigenvalues $\{\lambda_1 . \dots , \lambda_m\} \subseteq 2\pi\mathbb{Z}$ and $\sum_{i=1}^{m} (\#\text{ of eigenvectors for } \lambda_i) = n$
Let $\{v_1, \dots, v_n\}$ denote these linearly independent eigen vectors.
$\iff$ $\{v_1, \dots, v_n\}$ are eigenvectors for $\exp(M)$ and the eigenvalues for $\exp(M)$ are 1
$\iff$ $\exp(M)$ has $n$ Jordan blocks and eigenvalue 1.
$\iff$ $\exp(M) = I$